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On quantisation ambiguity†

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Abstract. The correspondence between classical and quantum theories is considered. We investigate the set of linear quantisations of classical Hamiltonians written in various canonical variables. The inverse problem of defining the classical limit of a quantum theory is also studied.

1. Introduction

The quantisation procedure is one of the basic notions one has to understand when working with quantum physics. Essentially, all existing quantum theories are constructed by quantisation of the corresponding classical theory.

To perform this procedure one has to address two issues. First, one has to establish the physical meaning of quantum variables by constructing the basis in the Hilbert space corresponding to the particular choice of the canonical variables x and p. Second, given a classical Hamiltonian $h(x, p)$ one has to construct the corresponding quantum Hamiltonian $\hat{H}$ governing the dynamics in Hilbert space.

In classical mechanics the state of a system is described by a point in phase space. The particular choice of the canonical variables (coordinates in the phase space) corresponds to some measuring apparatus. By changing the measuring apparatus we change a basis in the phase space. Mathematically this is achieved by means of a canonical transformation

$$x, p \xrightarrow{C} x', p'.$$

In quantum mechanics a state is described by a vector $|\psi\rangle$ in Hilbert space. The quantity $\langle x|\psi\rangle$ has the meaning of probability amplitude of measuring the classical quantity x. The transition to the basis $|x'\rangle$ corresponding to another classical variable x' is achieved by a unitary transformation

$$|x\rangle \xrightarrow{U} |x'\rangle.$$

Furthermore, the quantisation procedure is known to be not unique [1-5]. In [4] we considered a natural quantisation procedure defined by the requirement that the classical limit of the obtained quantum theory should coincide with its classical prototype. It was shown there that quantisations of the same classical theory expressed through different canonical variables are generally not equivalent. On the other hand, this natural quantisation procedure itself is not unique, since there is some freedom

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in the definition of the classical limit of a quantum theory [6] (see the discussion of this point in § 3).

Given a classical Hamiltonian $h(x, p)$ any quantisation procedure Q converts it to a quantum operator $\hat{H}$:

$$h(x, p) \xrightarrow{Q} \hat{H}.$$

For a given quantum Hamiltonian the procedure of taking its classical limit L is the operation inverse to quantisation:

$$\hat{H} \xrightarrow{L} h(x, p).$$

All the operations are schematically represented in figure 1. In this paper we analyse interrelations between these operations.

We shall restrict ourselves to the linear canonical transformations

$$\begin{aligned} x' &= Ax + Bp \\ p' &= Cx + Dp \end{aligned} \tag{1}$$

where the real numbers A, B, C, D satisfy

$$AD - BC = 1. \tag{2}$$

In quantum mechanics the corresponding unitary transformation is the Bogoliubov–Valatin transformation U [7]

$$\begin{aligned} \hat{x}' &= A\hat{x} + B\hat{p} = U\hat{x}U^{-1} \\ \hat{p}' &= C\hat{x} + D\hat{p} = U\hat{p}U^{-1}. \end{aligned} \tag{3}$$

The matrix U is given in appendix 1.

In § 2 we address the question of the uniqueness of the quantisation procedure. The set of all Hamiltonians obtained by any quantisation of a classical system described in terms of arbitrary canonical variables is found.

In § 3 the inverse problem is discussed: the uniqueness of the classical limit of a quantum theory is written in an arbitrary basis and, finally, in § 4 we discuss the general relation between quantum and classical mechanics.

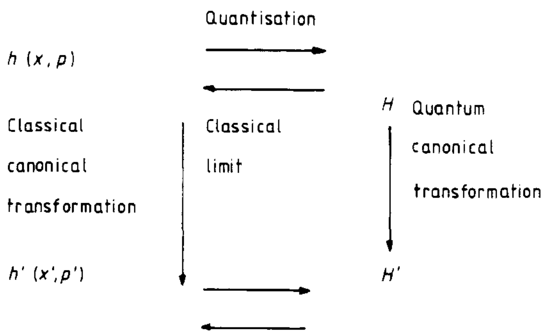

Figure 1. Relations between quantum and classical Hamiltonians.

2. Quantisation

For the construction of a quantum theory corresponding to some classical theory specified by the Hamiltonian $h(x, p)$ three procedures are usually used.

(i) Normal ordering procedure. One constructs annihilation and creation operators from a linear combination of $\hat{x}$ and $\hat{p}$:

$$\begin{aligned} \hat{a} &= (\omega/2\hbar)^{1/2}\hat{x} - [\mathrm{i}/(2\omega\hbar)^{1/2}]\hat{p} \\ \hat{a} &= (\omega/2\hbar)^{1/2}\hat{x} + [\mathrm{i}/(2\omega\hbar)^{1/2}]\hat{p} \end{aligned} \tag{4}$$

where ω is a positive constant. The operators $\hat{a}$ and $\hat{a}^+$ satisfy the commutation relation

$$[\hat{a}, \hat{a}^+] = 1. \tag{5}$$

Then one performs normal ordering of $h(x, p)$ with respect to $\hat{a}$ and $\hat{a}^+$. The result will be denoted by

$$\hat{H} =: h(x, p):_\omega. \tag{6}$$

This procedure obviously depends on ω. It was shown in [6] that the classical limit of $\hat{H}$, understood as the expectation value of $\hat{H}$ in eigenstates of $\hat{a}$, coincides with $h(x, p)$.

(ii) Symmetrisation procedure. In order to construct the Hermitian operator corresponding to the monomial classical expression

$$h(x, p) = x^n p^m \tag{7}$$

one can define the number of possible symmetrisations of x and p which are in general non-equivalent. For example, p^2x^3 can be quantised in either way:

$$\begin{aligned} \hat{H}_1 &= \tfrac{1}{2}(\hat{p}^2\hat{x}^3 + \hat{x}^3\hat{p}^2) \\ \hat{H}_2 &= \hat{p}\hat{x}^3\hat{p} \\ \hat{H}_3 &= \hat{x}\hat{p}\hat{x}\hat{p}\hat{x} \end{aligned} \tag{8}$$

and there exist other possibilities. For a sum of monomials the exact rule of symmetrisation should be specified for each term. The most popular is the Weyl symmetrisation [2].

The only cases in which symmetrisation is unambiguous are

$$h(x, p) = f(x) + g(p) + \alpha xp. \tag{9}$$

This is a situation in many cases of interest in quantum mechanics of few degrees of freedom, but there are also quantum theories which are not of this type †.

Generally, the classical limit for a quantum theory constructed in this way does not coincide with the classical Hamiltonian we started from ‡.

(iii) Path integral quantisation. Starting from the classical Hamiltonian $h(x, p)$ one constructs the generating functional [10, 11]

$$Z = \int \mathrm{d}x\, \mathrm{d}p\, \mathrm{e}^{\mathrm{i}S(x,p)} \tag{10}$$

† For example, a particle in a (varying) magnetic field, as well as some field theoretical models [8].

‡ The relation between the quantum Hamiltonian and the c-number function, as well as the correspondence between integrals of motion of classical and quantum theories in this quantisation prescription was studied by Hietarinta [9].

and directly derives from it quantum amplitudes. The path integral (10) is not defined in internal terms and its value depends on the particular limit process one adopts to calculate it [3, 5, 10]. For any ordering procedure we have a corresponding definition of path integral. This question, as well as quantisation by different symmetrisations, was studied in detail by Berezin [3] (see also [2]).

In this work we shall limit ourselves to the general normal ordering quantisation procedure. The most general operators linear in $\hat{x}$ and $\hat{p}$ and satisfying the commutation relations of equation (2) are not restricted to the one-parameter family defined by (1). All possible creation and annihilation operators are

$$\begin{aligned} \hat{a} &= \alpha\hat{x} - \mathrm{i}\beta\hat{p} \\ \hat{a}^+ &= \alpha^*\hat{x} + \mathrm{i}\beta^*\hat{p} \end{aligned} \tag{11}$$

where the complex numbers α and β are restricted by

$$\alpha\beta^* + \beta\alpha^* = 1/\hbar. \tag{12}$$

A convenient parametrisation of α and β is

$$\begin{aligned} \beta &= \mathrm{e}^{\mathrm{i}(\chi+\varphi)} 1/(2\omega \cos 2\varphi\hbar)^{1/2} \\ \alpha &= \mathrm{e}^{\mathrm{i}(\chi-\varphi)}[\omega/(2\cos 2\varphi\hbar)]^{1/2}. \end{aligned} \tag{13}$$

The matrix connecting $\hat{x}$, $\hat{p}$ with $\hat{a}$, $\hat{a}^+$ will be called the quantisation matrix Q

$$Q(\omega, \chi, \varphi) = \begin{pmatrix} \alpha & \mathrm{i}\beta \\ \alpha^* & -\mathrm{i}\beta^* \end{pmatrix}. \tag{14}$$

Performing normal ordering with respect to $\hat{a}$ and $\hat{a}^+$ we obtain a Hermitian operator $\hat{H}$.

For every classical Hamiltonian $h(x, p)$ we therefore define a three-parameter set of operators

$$\hat{H}_{\omega,\chi,\varphi} = :h(x, p):_{\omega,\chi,\varphi}. \tag{15}$$

These quantum Hamiltonians are generally not equivalent. However, one can show that quantum Hamiltonians corresponding to different parameters χ are connected by the unitary transformation U

$$\hat{H}_{\omega,\chi_1,\varphi} = U_{(\chi_2-\chi_1)}\hat{H}_{\omega\chi_2\varphi}U^{-1}_{(\chi_2-\chi_1)} \tag{16}$$

where U is the Bogoliubov-Valatin transformation [7]

$$U_{\chi_2-\chi_1} = \exp[\mathrm{i}(\chi_2-\chi_1)a^+a] = \exp\left(\frac{\mathrm{i}}{\hbar}\frac{(\chi_2-\chi_1)}{2\omega}(\hat{p}^2+\omega^2\hat{x}^2)\right). \tag{17}$$

On the other hand, the remaining two parameters ω and φ define non-equivalent Hamiltonians which are given (see appendix 3) by

$$H_{\omega\varphi} = \exp\left(\frac{\hbar}{4\omega\cos 2\varphi}\frac{\partial^2}{\partial x^2} - \frac{\omega\hbar}{4\cos 2\varphi}\frac{\partial^2}{\partial p^2} + \frac{\mathrm{i}\hbar\,\mathrm{e}^{-2\mathrm{i}\varphi}}{2\cos 2\varphi}\frac{\partial}{\partial x}\frac{\partial}{\partial p}\right)h(\hat{x}, \hat{p}) \tag{18}$$

where $h(\hat{x}, \hat{p})$ has the same functional form as the classical Hamiltonian with all $\hat{p}$ written to the right of $\hat{x}$.

In classical mechanics the same dynamics can be described in different canonical variables. Furthermore, as we discussed earlier, for every choice of canonical variables there are different possibilities for quantising it. We now investigate the whole set of quantum Hamiltonians obtained in this way.

The linear canonical transformations given by equation (3) constitute the three-parameter group SL($2R$) which we shall call the classical Bogoliubov–Valatin group. This group is studied in appendix 1 where the physical meaning of its parameters λ, θ, σ is discussed.

A given classical dynamics may be described by any one of the Hamiltonians

$$h^{\lambda\theta\sigma}(x, p) = h(x'(x, p), p'(x, p)). \tag{19}$$

Each of them can be quantised by normal ordering with respect to any creation and annhilation operators defined by equation (10):

$$\hat{H}^{\lambda\theta\sigma}_{\omega\chi\varphi} = :h^{\lambda\theta\sigma}(x, p):_{\omega\chi\varphi}. \tag{20}$$

Thus we have a six-parameter set of quantum operators. However some of them are equivalent, i.e. connected by a unitary transformation. Indeed, the Hamiltonians $\hat{H}^{\lambda\theta\sigma}_{\omega\chi\varphi}$ and $\hat{H}^{\lambda'\theta'\sigma'}_{\omega'\chi'\varphi'}$ are identical provided

$$Q(\omega, \chi, \varphi)C^{-1}(\lambda, \theta, \sigma) = Q(\omega', \chi', \varphi')C^{-1}(\lambda', \theta', \sigma') \tag{21}$$

where the matrix C specifying a canonical transformation is defined in appendix 1. This can be readily seen, since annihilation operators $\hat{a}$ and $\widehat{a'}$ defined by

$$\begin{aligned} \hat{a} &= Q(\omega, \chi, \varphi)\begin{pmatrix} x \\ p \end{pmatrix} \\ \hat{a}' &= Q(\omega', \chi', \varphi')\begin{pmatrix} x' \\ p' \end{pmatrix} \end{aligned} \tag{22}$$

are identical if

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = C(\lambda', \theta', \sigma')C^{-1}(\lambda, \theta, \sigma)\begin{pmatrix} x \\ p \end{pmatrix}. \tag{23}$$

Therefore different quantum Hamiltonians constitute a three-parameter set which can be parametrised by the quantisation matrix only. Furthermore, it follows from equation (14) that the non-equivalent Hamiltonians constitute a three-parameter set only. We observe that this set is independent of the canonical representation of the classical dynamics. The explicit example of this set for the anharmonic oscillator is given in appendix 3.

3. The classical limit of quantum theory

Up to now we have dealt with quantisation of a classical theory. Now we address the inverse problem of finding a classical limit for a given quantum theory. The definition of the classical limit we adopt here is that introduced by Klauder and others [6, 12].

In this definition coherent states play a crucial role. The classical limit Hamiltonian $h(x, p)$ is defined as the expectation value of the operator $\hat{H}$ on the set of coherent states $|x, p)$

$$h(x, p) = (x, p|\hat{H}|x, p) \tag{24}$$

where $|x, p)$ denotes the coherent state that has expectation values

$$\begin{aligned}(x, p|\hat{x}|x, p) &= x\\ (x, p|\hat{p}|x, p) &= p.\end{aligned} \tag{25}$$

However, the definition of the coherent states is not unique. Indeed, a coherent state is defined as an eigenstate of the annihilation operator $\hat{a}$

$$\hat{a}|x, p) = a|x, p) \tag{26}$$

where a is a complex number. These states constitute an overcomplete set [6]. But, as was mentioned in § 2, the annihilation operator $\hat{a}$ can be defined by equation (3), which contains three arbitrary parameters ω, φ and χ. Therefore one expects that there is a system of coherent states for each choice of the parameters ω, φ, χ.

These states in the configuration space are (see appendix 2)

$$\langle x|q, p)_{\omega,\varphi,\chi} = \text{constant} \times \exp\left[-\frac{1}{2}\frac{\omega}{\hbar}e^{-2i\varphi}\left(x - q - \frac{ip}{\hbar}\right)^2\right]. \tag{27}$$

We observe that the parameter χ does not influence the structure of coherent states, while ω and φ are essential. If the functions $h^{\omega\varphi}(q, p)$ and $h^{\omega_0\varphi_0}(q, p)$ are viewed as classical Hamiltonians they describe different classical systems since they cannot be connected by a canonical transformation.

The situation closely resembles that of quantisation where we had a two-parameter set of non-equivalent quantum Hamiltonians corresponding to one classical dynamics.

We pursue further the analogy with the quantisation procedure. In appendix 1 we define the three-parameter group of Bogoliubov-Valatin transformations SL(2, R). The quantum analogue of the set of classical Hamiltonians written in different (linear) canonical variables is the set of all Hamiltonians

$$\hat{H}_{\lambda\theta\sigma} = U_{\lambda\theta\sigma}\hat{H}U^{-1}_{\lambda\theta\sigma}. \tag{28}$$

For any such Hamiltonian we define classical limits via

$$h^{\omega\varphi\chi}_{\lambda\theta\sigma} = {}_{\omega\varphi\chi}(q, p|U_{\lambda\theta\sigma}HU^{-1}_{\lambda\theta\sigma}|q, p)_{\omega\varphi\chi}. \tag{29}$$

However, as in the case of quantisation, the classical Hamiltonians $h^{\omega\varphi\chi}_{\lambda\varphi\sigma}$ and $h^{\omega'\varphi'\chi'}_{\lambda'\theta'\sigma'}$ are identical provided

$$Q(\omega, \chi, \varphi)C^{-1}(\lambda, \theta, \sigma) = Q(\omega', \chi', \varphi')C^{-1}(\lambda', \theta', \sigma'). \tag{30}$$

Therefore there is a three-parameter set of non-identical classical Hamiltonians that can be represented by $h^{\omega\varphi\chi}$. In fact, as we observed earlier, Hamiltonians with different χ coincide and thus it is a two-parameter set only.

We notice that in general those classical Hamiltonians depend on the Planck constant $\hbar$. It follows from (A3) that only one of them is $\hbar$ independent. This is the one obtained as the classical limit with parameters (ω, φ) equal to (ω_0, φ_0)—the parameters of the quantisation matrix used for quantising the theory.

4. Discussion

We have considered some general aspects of the relation between a classical and a corresponding quantum mechanics. Two procedures inverse one to another were studied:

(1) quantisation of a classical theory via normal ordering, and

(2) taking the classical limit of a quantum mechanics.

It is known that there is an ambiguity in the quantisation of a classical theory. However, the ambiguity that usually deserves the greatest attention is the one between different orderings of the operators $\hat{x}$ and $\hat{p}$. The ambiguity of quantisation via different normal orderings is much wider. For example, in quantising the Hamiltonian

$$f(x)+g(p)$$

orderings of $\hat{x}$ and $\hat{p}$ are inessential, but different normal orderings definitely give different quantum theories.

It was shown that different quantisations of a classical Hamiltonian $h(x,p)$ lead to the two-parameter set of non-equivalent quantum operators $\hat{H}_{\omega,\varphi}$. This set happens to be independent of the choice of classical canonical variables as long as they are connected by a linear canonical transformation. We have found explicit expressions for all quantum Hamiltonians that may be obtained from given classical Hamiltonians via different normal ordering procedures as well as for all classical limits of a given quantum Hamiltonian.

The ambiguity in defining the classical limit is similar. By taking the classical limit of a quantum Hamiltonian $\hat{H}$ one obtains the two-parameter set of functions $h_{\omega\varphi}(x, p)$ which cannot be connected by a classical canonical transformation. This set is also independent of the choice of a basis in Hilbert space.

In view of these correspondences one can conclude that there is a one-to-one correspondence between the set of all classical dynamics and the set of all quantum dynamics. This conclusion, at first sight, might seem a little surprising.

A classical state is described by a point (x, p) in the two-dimensional phase space, while a quantum state is a vector in the infinite-dimensional Hilbert space. Any possible classical dynamics can be specified by a real function on the phase space $h(x, p)$. One might think that in the Hilbert space there is much more room for possible dynamics since, in principle, they could be described by any operator on Hilbert space.

However, quantum mechanics poses extremely strong restrictions on the possible evolution of a quantum state: the superposition principle. This principle implies that the operator $\hat{H}$ should be linear and the set of all linear operators on the huge Hilbert space is in one-to-one correspondence with the set of all functions on the phase space.

Acknowledgment

We are very grateful to Professor L P Horwitz for fruitful discussions.

Appendix 1. Classical and quantum Bogoliubov–Valatin transformations

Given the classical canonical Bogoliubov-Valatin transformation

$$\begin{aligned} x' &= Ax + Bp \\ p' &= Cx + Dp \end{aligned} \tag{A1.1}$$

one can find a quantum unitary transformation U such that

$$\begin{aligned} \hat{x}' &= A\hat{x} + B\hat{p} = U\hat{x}U^{-1} \\ \hat{p}' &= C\hat{x} + D\hat{p} = U\hat{p}U^{-1}. \end{aligned} \tag{A1.2}$$

This transformation is well known [7] and has the general form

$$U = \exp[\alpha\hat{x}^2 + \beta(\hat{x}\hat{p} + \hat{p}\hat{x}) + \gamma\hat{p}^2]. \tag{A1.3}$$

In order to find coefficients corresponding to the transformation (A1.1) let us parametrise and investigate the group of transformations (A1.1).

The group contains the following physically meaningful one-parameter subgroups.

(i) Rescalings:

$$x' = \lambda x \qquad p' = (1/\lambda)p. \tag{A1.4}$$

(ii) Phase space rotations:

$$\begin{aligned} x' &= \cos(\theta)x + (1/\hbar)\sin(\theta)p \\ p' &= -\hbar\sin(\theta)x + \cos(\theta)p \end{aligned} \tag{A1.5}$$

where $\hbar$ is a dimensional constant and θ is the parameter of the rotation:

(iii) Classical gauge transformation:

$$x' = x \qquad p' = p + A(x) \qquad A(x) = \sigma x. \tag{A1.6}$$

A general classical Bogoliubov-Valatin transformation can be conveniently parametrised as follows:

$$C^{\lambda,\theta,\sigma} = C(\lambda)C(\theta)C(\sigma) \tag{A1.7}$$

where C is the matrix of the linear transformation (A1.1).

It can be easily verified that the quantum counterparts of these subgroups (the corresponding quantum Bogoliubov-Valatin transformations) are the following.

(i) Rescaling subgroup:

$$U = \exp[-(\mathrm{i}/2\hbar)\ln\lambda(\hat{x}\hat{p} + \hat{p}\hat{x})]. \tag{A1.8}$$

(ii) Phase space rotations:

$$U = \exp[-(\mathrm{i}/2\hbar)\theta(\hat{p}^2/k + k\hat{x}^2)]. \tag{A1.9}$$

(iii) Gauge transformations:

$$U = \exp[-(\mathrm{i}/2\hbar)\sigma\hat{x}^2]. \tag{A1.10}$$

Any element of the SL$(2R)$ group can be obtained as a product of these transformations, as in equation (A1.7).

Appendix 2. Coherent states

In this appendix we give the explicit expression for an eigenstate of the annihilation operator $\hat{a}$:

$$\hat{a} = \alpha\hat{x} - \mathrm{i}\beta\hat{p} \tag{A2.1}$$

where the complex numbers α and β can be written as

$$\begin{aligned} \alpha &= \mathrm{e}^{\mathrm{i}(\chi-\varphi)}[\omega/(2\cos 2\varphi\hbar)]^{1/2} \\ \beta &= \mathrm{e}^{\mathrm{i}(\chi+\varphi)}/(2\hbar\cos 2\varphi)^{1/2}. \end{aligned} \tag{A2.2}$$

These states in coordinate representation satisfy the following differential equation:

$$(\alpha x + \beta\partial/\partial x)|q, p) = (\alpha q - \mathrm{i}\beta p)|q, p) \tag{A2.3}$$

where q and p are the expectation values of the operators $\hat{x}$ and $\hat{p}$, respectively.

The solution of equation (A2.3) has the form

$$|q, p) = N\exp\left[-\frac{\omega}{2\hbar}\mathrm{e}^{-2\mathrm{i}\varphi}\left(x - q + \frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}2\varphi}}{\omega\hbar}p\right)^2\right]. \tag{A2.4}$$

The normalisation condition implies

$$N = \left(\frac{\omega\cos 2\varphi}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\cos 2\varphi}{2\omega\hbar}p^2\right). \tag{A2.5}$$

Appendix 3. Explicit expression for a quantised Hamiltonian

The expectation value of the operator $\hat{x}^n\hat{p}^m$ in the Gaussian state

$$\psi(x) = \left(\frac{\omega\cos 2\varphi}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{\cos 2\varphi}{2\omega\hbar}p^2\right)\exp\left(-\frac{A}{2}(x-z)^2\right) \tag{A3.1}$$

where

$$A = (\omega/\hbar)\,\mathrm{e}^{-2\mathrm{i}\varphi} \qquad z = q - \mathrm{i}(\mathrm{e}^{\mathrm{i}2\varphi}/\omega\hbar)p \tag{A3.2}$$

is

$$\begin{aligned} \langle\hat{x}^n\hat{p}^m\rangle = N^2\int & \mathrm{d}x\,\exp[-\tfrac{1}{2}A^*(x-z^*)^2]x^n \\ & \times(\mathrm{i}\sqrt{\tfrac{1}{2}A})^m H_m(\sqrt{\tfrac{1}{2}A}(x-z))\exp[-\tfrac{1}{2}A(x-z)^2]. \end{aligned} \tag{A3.3}$$

The function $H_m(y)$ is Hermite polynomial of order m. It can be easily verified that the same expression can be obtained by taking derivatives with respect to parameters:

$$\begin{aligned} \langle\hat{x}^n\hat{p}^m\rangle &= N^2\frac{1}{(A+A^*)^n}\left(\mathrm{i}\frac{\partial}{\partial z}\right)^m\exp\left(-\frac{1}{2(A+A^*)}(Az+A^*z^*)^2\right)\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial z^*}\right)^n \\ &\quad\times\int\exp\left(-\frac{A^*}{2}(x-z^*)^2-\frac{A}{2}(x-z)^2+\frac{1}{2(A+A^*)}(Az+A^*z^*)^2\right)\mathrm{d}x \\ &= N^2\left(\frac{1}{A+A^*}\right)^n\left(\mathrm{i}\frac{\partial}{\partial z}\right)^m\exp\left(-\frac{1}{2(A+A^*)}(Az+A^*z^*)^2\right)\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial z^*}\right)^n \\ &\quad\times\exp\left(\frac{A+A^*}{2}\left(\frac{1}{\omega^2}p^2+x^2\right)\right). \end{aligned} \tag{A3.4}$$

With the help of (A2.4) this can be written as:

$$\langle\hat{x}^n\hat{p}^m\rangle = \mathrm{i}^m(2\omega\hbar\cos 2\varphi)^{-n}[-\mathrm{i}(\omega\cos 2\varphi)^{1/2}]^n \sum_{k=0}^{m} C_m^k\left(\frac{1}{2}\frac{\mathrm{e}^{-\mathrm{i}2\varphi}}{\hbar\cos 2\varphi}\right)^k\left(\frac{-\mathrm{i}\omega}{2\hbar\cos 2\varphi}\right)^{m-k}$$
$$\times[2\mathrm{i}(\omega\cos 2\varphi)^{1/2}\hbar]^k\frac{n!}{(n-k)!}\left[\mathrm{i}\left(\frac{\cos 2\varphi}{\omega\hbar}\right)^{1/2}\right]^{m-k}$$
$$\times H_{n-k}\left(\mathrm{i}(\omega\cos 2\varphi\hbar)^{1/2}q\right)H_{m-k}\left[\mathrm{i}\left(\frac{\cos 2\varphi}{\hbar\omega}\right)^{1/2}p\right]. \tag{A3.5}$$

Because of the identity

$$H_n(\mathrm{i}ax) = \mathrm{e}^{(1/2na^2)(\partial^2/\partial x^2)}x^n(2\mathrm{i}a)^n \tag{A3.6}$$

the above, finally, can be rewritten as

$$\langle\hat{x}^n\hat{p}^m\rangle = \exp\left(\frac{\hbar}{4\omega\cos 2\varphi}\frac{\partial^2}{\partial q^2}+\frac{\hbar\omega}{4\cos 2\varphi}\frac{\partial^2}{\partial p^2}-\frac{\hbar\mathrm{e}^{-2\mathrm{i}\varphi}}{\mathrm{i}2\cos 2\varphi}\frac{\partial}{\partial p}\frac{\partial}{\partial q}\right)q^n p^m. \tag{A3.7}$$

Since any 'physical' function can be viewed as a sum of monomials formula (A3.7) implies two following consequences.

First, we can write down in closed form a quantum Hamiltonian obtained by quantisation of $h(x, p)$ with the quantisation matrix $Q(\omega_0, \varphi_0)$:

$$\hat{H} = \exp\left(-\frac{\hbar}{4\omega_0\cos 2\varphi_0}\frac{\partial^2}{\partial x^2}-\frac{\omega_0\hbar}{4\cos 2\varphi_0}\frac{\partial^2}{\partial p^2}+\frac{\hbar\,\mathrm{e}^{-2\mathrm{i}\varphi_0}}{\mathrm{i}2\cos 2\varphi_0}\frac{\partial}{\partial p}\frac{\partial}{\partial x}\right)h(x, p)\bigg|_{x=\hat{x},p=\hat{p}} \tag{A3.8}$$

where all operators $\hat{p}$ are written to the right of the operators $\hat{x}$.

Second, one can write the classical limit of $\hat{H}$ with respect to coherent states with different parameters ω, φ:

$$h_{\omega\varphi}(q, p) = {}_{\omega\varphi}\langle q, p|:h(x, p):_{\omega_0\varphi_0}|q, p\rangle_{\omega\varphi}$$
$$= \exp\left[\left(\frac{\hbar}{4\omega\cos 2\varphi}-\frac{\hbar}{4\omega_0\cos 2\varphi_0}\right)\frac{\partial^2}{\partial q^2}+\hbar\left(\frac{\omega}{4\cos 2\varphi}-\frac{\omega_0}{4\cos 2\varphi_0}\right)\frac{\partial^2}{\partial p^2}\right.$$
$$\left.-\frac{\hbar}{2\mathrm{i}}\left(\frac{\mathrm{e}^{-2\mathrm{i}\varphi_0}}{\cos 2\varphi}-\frac{\mathrm{e}^{-2\mathrm{i}\varphi_0}}{\cos 2\varphi_0}\right)\frac{\partial}{\partial p}\frac{\partial}{\partial q}\right]h(q, p)+\mathrm{cc}. \tag{A3.9}$$

To exemplify these general expressions we consider specifically the case of the anharmonic oscillator:

$$h(q, p) = \tfrac{1}{2}p^2+\tfrac{1}{2}\Omega^2q^2+(\lambda/4!)q^4. \tag{A3.10}$$

According to (A3.8) the set of quantum Hamiltonians obtained by the general normal ordering procedure is

$$:h(\hat{x}, p):_{\omega_0\varphi_0=}\tfrac{1}{2}\hat{p}^2+\frac{1}{2}\left(\Omega^2-\frac{\hbar\lambda}{4\omega_0\cos 2\varphi_0}\right)\hat{x}^2+\frac{\lambda}{4!}\hat{x}^4$$
$$-\hbar\left(\frac{\Omega^2}{4\omega_0\cos 2\varphi_0}+\frac{\omega_0}{4\cos 2\varphi_0}\right)+\hbar^2\frac{\lambda}{32(\omega_0\cos 2\varphi_0)^2}. \tag{A3.11}$$

The set of all classical limits of these Hamiltonians is

$$h_{\omega\varphi}(q,p)=\tfrac{1}{2}p^2+\frac{1}{2}\left[\Omega^2+\hbar\frac{\lambda}{4}\left(\frac{1}{\omega\cos 2\varphi}-\frac{1}{\omega_0\cos 2\varphi_0}\right)\right]q^2$$
$$+\frac{\lambda}{4!}q^4+\frac{\hbar}{4}\left[\frac{\omega}{\cos 2\varphi_0}-\frac{\omega_0}{\cos 2\varphi_0}+\frac{\Omega^2}{\omega\cos 2\varphi}-\frac{\Omega^2}{\omega_0\cos 2\varphi_0}\right]$$
$$+\hbar^2\frac{\lambda}{32}\left(\frac{1}{\omega\cos 2\varphi}-\frac{1}{\omega_0\cos 2\varphi_0}\right)^2. \tag{A3.12}$$

Observe that only if $\omega=\omega_0$ and $\varphi=\varphi_0$ is $h_{\omega\varphi}(q,p)$ independent of the Planck constant $\hbar$, and then, of course,

$$h_{\omega_0\varphi_0}(q,p)=h(q,p). \tag{A3.13}$$

This is a general feature and does not depend on the specific form of $h(q,p)$.

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